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Scattering Theory in Quantum Field Theory in Configuration Space

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The recent analysis of the propagation of relativistic particles in *spacetime* and their localization problem is used to develop scattering theory in quantum field theory in *configuration* space. An explicit functional expression is derived for the underlying transition amplitudes having a consistent probabilistic interpretation. Some of the basic ingredients in the analysis are the functional approach developed earlier for transition amplitudes and the amplitudes for stimulated emission of particles by external sources in spacetime.

1. INTRODUCTION

In describing the scattering of relativistic particles in quantum field theory one almost always carries out the analysis in momentum space. The many experiments (e.g., Franson and Potocki, 1988; Grangier *et al.*, 1986, and the many references therein) indicating the localization, by detectors, of not only massive but also massless particles has urged us to develop scattering theory in configuration space. One is eventually interested in asking probabilistic questions, such as, What is the probability that a given particle emerges spatially within a cone after a collision or some decay process? Earlier efforts (e.g., Han *et al.*, 1987; Ali, 1985 and the many references therein) dealing with the construction of position operators and wavefunctions (as done in nonrelativistic theory) are so remote from the field theory formulation of particle dynamics that we have reconsidered the localization problem afresh (Manoukian, 1989, and references therein) directly from field theory and an underlying *unitarity* expansion carried out in configuration space having a consistent probabilistic interpretation.

The purpose of this work is to develop the scattering theory of relativistic particles in quantum field theory in configuration space, based on my earlier

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analysis (Manoukian, 1989) of their propagation in spacetime. An explicit expression for such amplitudes is derived as an extension of the investigation (Manoukian, 1988) dealing with a functional approach to transition amplitudes in momentum space. I make a systematic use of the analysis of stimulated emission of particles by external sources in spacetime (Manoukian, 1989). In this paper no specific models are considered and it is restricted to a real, massive scalar field. This work, however, lays the foundation for more general studies involving physically more relevant theories. The more realistic gauge theories (e.g., Manoukian, 1986) are beyond the scope of the present paper and, together with applications, will be considered in a subsequent report.

2. SCATTERING THEORY IN CONFIGURATION SPACE

Consider a massive scalar field $\phi(x)$ with interaction Lagrangian density $L_I(\phi)$. Also couple the field linearly to an external (c-number) source $K(x)$. Scattering in and out states are denoted by $|g-\rangle$ and $|f+\rangle$, respectively.

The quantum action principle (Schwinger, 1951, 1954; Lam, 1965; Manoukian, 1985) gives the expression

$$\langle f+|g-\rangle = \exp\left[i \int (dx)\, L_I\left(-i\frac{\delta}{\delta K(x)}\right)\right]\langle f+|g-\rangle^{K,0}|_{K=0} \qquad (1)$$

where $\langle f+|g-\rangle^{K,0}$ is the transition amplitude with L_I set equal to zero and in the presence of an intervening source $K(x)$ localized in time. The latter amplitude is precisely the amplitude for stimulated emission by an external source $K(x)$ considered earlier (Manoukian, 1989) where there is initially a given number of particles before the intervening source $K(x)$ is switched on.

I spell out the expression for $\langle f+|g-\rangle^{K,0}$. To this end, I consider a discrete (Schwinger, 1970; Manoukian, 1984) variable notation (a lattice) for space, at any given time, $\{\mathbf{y}_1, \mathbf{y}_2, \ldots\}$. Let y_1^0 and y_2^0 be any given times such that $K(x)=0$ for $x^0 \leq y_1^0$ and for $x^0 \geq y_2^0$. That is, the intervening source is switched on after a time reading y_1^0 and it is then switched off before a time reading y_2^0. Let M_1 denote the number of particles localized at $\mathbf{y}_1$, M_2 the number of particles localized at $\mathbf{y}_2$, and so on, *before* the intervening source $K(x)$ is switched on. Similarly, let N_1 denote the number of particles localized at $\mathbf{y}_1$, N_2 the number of particles localized at $\mathbf{y}_2$, and so on, *after* the intervening source $K(x)$ is finally switched off. Then (Manoukian, 1989)

$$\langle f+|g-\rangle^{K,0} = \lim_{\substack{y_2^0 \to +\infty \\ y_1^0 \to -\infty}} \langle N; N_1, N_2, \ldots, y_2^0 | M; M_1, M_2, \ldots, y_1^0\rangle^{K,0} \qquad (2)$$

where

$$
\begin{aligned}
&\langle N; N_1, N_2, \ldots, y_2^0 | M; M_1, M_2, \ldots, y_1^0\rangle^{K,0} \\
&\quad = (N_1!\, N_2! \ldots M_1!\, M_2! \ldots)^{1/2} \\
&\quad\quad \times \sum_* \frac{(ia_1)^{N_1-m_1}}{(N_1-m_1)!} \frac{(ia_2)^{N_2-m_2}}{(N_2-m_2)!} \cdots \frac{(\tilde{\delta}_{11})^{m_{11}}}{m_{11}!} \frac{(\tilde{\delta}_{12})^{m_{12}}}{m_{12}!} \cdots \\
&\quad\quad \times \frac{(\tilde{\delta}_{21})^{m_{21}}}{m_{21}!} \frac{(\tilde{\delta}_{22})^{m_{22}}}{m_{22}!} \cdots \\
&\quad\quad \times \frac{(ia_1^*)^{M_1-\sum_i m_{i1}}}{(M_1-\sum_i m_{i1})!} \frac{(ia_2^*)^{M_2-\sum_i m_{i2}}}{(M_2-\sum_i m_{i2})!} \cdots \langle 0_+ | 0_-\rangle^{K,0}
\end{aligned}
\tag{3}
$$

$\sum_*$ stands for a summation over all nonnegative integers: $N_1, N_2, \ldots, M_1, M_2, \ldots; m_1, m_2, \ldots; m_{11}, m_{12}, \ldots; m_{21}, \ldots$; satisfying the constraints

$$
\begin{aligned}
N_1 + N_2 + \cdots &= N, & M_1 + M_2 + \cdots &= M \\
m_{11} + m_{12} + \cdots &= m_1, & 0 \le m_{11} + m_{21} + \cdots &\le M_1 \\
m_{21} + m_{22} + \cdots &= m_2, & 0 \le m_{12} + m_{22} + \cdots &\le M_2 \\
&\vdots & &\vdots \\
& & 0 \le m_1 \le N_1 & \\
& & 0 \le m_2 \le N_2 & \\
& & \vdots &
\end{aligned}
\tag{4}
$$

The a_i are explicitly given by

$$
\begin{aligned}
a_i &= (d^3\mathbf{y}_i)^{1/2} a(y_i) \\
a(y) &= \int \frac{d^3\mathbf{k}}{(2\pi)^3 (2k^0)^{1/2}}\, e^{iky} K(k), \qquad k^0 = (\mathbf{k}^2 + m^2)^{1/2}
\end{aligned}
\tag{5}
$$

$$
K(k) = \int (dx)\, e^{-ikx} K(x) \tag{6}
$$

$a(y)$ may be also rewritten as

$$
a(y) = \int (dx)\, \Delta(y-x) K(x) \tag{7}
$$

$$
\Delta(y-x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 (2k^0)^{1/2}}\, e^{ik(y-x)} \tag{8}
$$

$\langle 0_+|0_-\rangle^{K,0}$ is the well-known expression for the vacuum-to-vacuum transition amplitude in the absence of the nonlinear coupling L_I:

$$\langle 0_+|0_-\rangle^{K,0} = \exp\left[\frac{i}{2}\int (dx)\,(dx')\,K(x)\Delta_+(x-x')K(x')\right] \tag{9}$$

$$\Delta_+(x-x') = i\int \frac{d^3\mathbf{k}}{(2\pi)^3 2k^0}\,e^{ik(x-x')}, \qquad k^0 = (\mathbf{k}^2+m^2)^{1/2}, \quad \text{for} \quad x^0 > x'^0 \tag{10}$$

Finally,

$$\tilde{\delta}_{ij} = (d^3\mathbf{y}_i\, d^3\mathbf{y}_j)^{1/2}\tilde{\delta}(y_i - y_j) \tag{11}$$

$$\tilde{\delta}(y-y') = \int d^3\mathbf{x}\,\Delta(y-x)\,i\frac{\overleftrightarrow{\partial}}{\partial x^0}\,\Delta(x-y') \tag{12}$$

$y^0 > x^0 > y'^0$ denotes the amplitude that a particle propagates from time-space coordinate $y' = (y'^0, \mathbf{y}')$ to time-space coordinate $y = (y^0, \mathbf{y})$. Note that (12) does not coincide with the so-called familiar Feynman propagator $\Delta_+(y-y')$ in (10).

If we denote the transition amplitude by

$$\langle f_+|g_-\rangle = \langle N;\, N_1, N_2, \ldots, +|M;\, M_1, M_2, \ldots, -\rangle \tag{13}$$

with the actual nonlinear coupling, in the absence of the external source, then we finally obtain for the transition amplitude of M particles, M_1 of which are initially at $\mathbf{y}_1$, M_2 of which are initially at $\mathbf{y}_2$, and so on, to N particles, N_1 of which are finally at $\mathbf{y}_1$, N_2 of which are finally at $\mathbf{y}_2$, and so on, is, from (1),

$$\begin{aligned}\langle N;\, &N_1, N_2, \ldots, +|M;\, M_1, M_2, \ldots, -\rangle \\ &= \exp\left[i\int (dx) L_I\left(-i\frac{\delta}{\delta K(x)}\right)\right] \\ &\quad \times \langle N;\, N_1, N_2, \ldots, y_2^0|M;\, M_1, M_2, \ldots, y_1^0\rangle^{K,0}|_{K=0}\end{aligned} \tag{14}$$

for $y_2^0 \to +\infty$, $y_1^0 \to -\infty$, with the amplitude $\langle\cdot|\cdot\rangle^{K,0}$ on the right-hand side of (14) defined through (3)-(12). Equation (14) is the main result of the paper. In the next section I consider so-called connected amplitudes and further simplifications, and finally write the amplitudes in an LSZ form.

3. FURTHER CONSIDERATIONS

I consider a simplification of the very general expression in (14). To this end, consider the scattering of r particles ($M = r$) with nonoverlapping positions $\mathbf{y}_1, \ldots, \mathbf{y}_r$ to s particles $(N = s)$ with nonoverlapping positions $\mathbf{y}_{r+1}, \ldots, \mathbf{y}_{r+s}$. Hence, in (4), $M_1 = 1, \ldots, M_r = 1$, $N_{r+1} = 1, \ldots, N_{r+s} = 1$, and all the other M_i, N_j are zero. Let us also consider connected amplitudes. Clearly any such amplitude excludes any term involving the amplitudes in (12) which deal with particles not participating in the scattering process. From (3) and (4) we then have

$$\begin{aligned}\langle s; 1_{r+1}, \ldots, 1_{r+s}; y_2^0 | r; 1_1, 1_2, \ldots, 1_r; y_1^0 \rangle_c^{K,0} \\ = (ia_1) \ldots (ia_r)(ia_{r+1}^*) \ldots (ia_{r+s}^*) \langle 0_+ | 0_- \rangle^{K,0}\end{aligned} \tag{15}$$

Hence, upon writing

$$\begin{aligned}&\langle s; 1_{r+1}, \ldots, 1_{r+s}, + | r; 1_1, 1_2, \ldots, 1_r, - \rangle_c \\ &= \prod_{i=1}^{r+s} (d^3\mathbf{y}_i)^{1/2} (s; 1_{r+1}, \ldots, 1_{r+s}, + | r; 1_1, 1_2, \ldots, 1_r, -)_c\end{aligned} \tag{16}$$

we have from (14)

$$\begin{aligned}&(s; 1_{r+1}, 1_{r+2}, \ldots, 1_{r+s}, + | r; 1_1, 1_2, \ldots, 1_r, -)_c \\ &= (i)^{r+s} \left(\prod_{i=1}^{r} \int (dx_i)\, \Delta(y_i - x_i) \right) \left(\prod_{j=r+1}^{r+s} \int (dx_i)\, \Delta^*(y_j - x_j) \right) \\ &\quad \times \exp\left[i \int (dx)\, L_I\left(-i \frac{\delta}{\delta K(x)} \right) \right] K(x_1) \ldots K(x_{r+s}) \langle 0_+ | 0_- \rangle^{K,0} \Big|_{K=0}^{c}\end{aligned} \tag{17}$$

where $y_i^0 = y_2^0 \to +\infty$, for $i = r+1, \ldots, r+s$, and $y_i^0 = y_1^0 \to -\infty$, for $i = 1, \ldots, r$. To simplify (17) further, we may use the identity

$$\begin{aligned}&\exp\left[i \int (dx)\, L_I\left(-i \frac{\delta}{\delta K(x)} \right) \right] K(y) \\ &= \left[K(y) + L_I'\left(-i \frac{\delta}{\delta K(y)} \right) \right] \exp\left[i \int (dx)\, L_I\left(-i \frac{\delta}{\delta K(x)} \right) \right]\end{aligned} \tag{18}$$

with $L_I'(z) = (d/dz)\, L_I(z)$, and the functional differential equation

$$(-\Box^2 + m^2)(-i) \frac{\delta}{\delta K(x)} \langle 0_+ | 0_- \rangle^K = \left[K(x) + L_I'\left(-i \frac{\delta}{\delta K(x)} \right) \right] \langle 0_+ | 0_- \rangle^K \tag{19}$$

where $\langle 0_+ | 0_- \rangle^K$ is the exact vacuum-to-vacuum transition amplitude. Hence we may rewrite the right-hand side of (17) in an LSZ form:

$$\begin{aligned}&(i)^{r+s} \left(\prod_{i=1}^{r} \int (dx_i)\, \Delta(y_i - x_i) \right) \left(\prod_{j=r+1}^{r+s} \int (dx_j)\, \Delta^*(y_j - x_j) \right) \\ &\quad \times \left(\prod_{i=1}^{r+s} (-\Box_i^2 + m^2) \right) \tau_c(x_1, \ldots, x_{r+s})\end{aligned} \tag{20}$$

where

$$\tau(x_1, \ldots, x_{r+s}) = \left(\prod_{j=1}^{r+s} (-i) \frac{\delta}{\delta K(x_j)} \right) \langle 0_+ | 0_- \rangle^K |_{K=0} \tag{21}$$

The Δ are defined in (8). Equation (20) is a special case of the general expression derived in (14), (3)–(12). Extension of the present work to the more realistic gauge theories, together with applications, will be considered in a future report.

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